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EQUIVARIANT QUANTUM COHOMOLOGY OF COTANGENT BUNDLE OF G/P

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Abstract. Let G denote a complex semisimple linear algebraic group, P a parabolic subgroup of G and $\mathcal{P} = G/P$. We identify the quantum multiplication by divisors in $T^*\mathcal{P}$ in terms of stable basis, which is introduced in [9]. Using this and the restriction formula for stable basis ([17]), we show that the $G \times \mathbb{C}^*$-equivariant quantum multiplication formula in $T^*\mathcal{P}$ is conjugate to the formula conjectured by Braverman.

1. Introduction

The main goal of this paper is to study the equivariant quantum cohomology of $T^*\mathcal{P}$, which is a special case of symplectic resolutions. Recall from [7] that a smooth algebraic variety X with a holomorphic symplectic form ω is called a symplectic resolution if the affinization map

$$X \to X_0 = \operatorname{Spec} H^0(X, \mathcal{O}_X)$$

is projective and birational. Conjecturally all the symplectic resolutions of the form T^*M for a smooth algebraic variety M are of the form $T^*\mathcal{P}$, see [7]. In [3], Fu proved that every symplectic resolution of a normalization of a nilpotent orbit closure in a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to $T^*\mathcal{P}$ for some parabolic subgroup P in G.

In [9], Maulik and Okounkov defined the stable basis for a wide class of varieties, which include symplectic resolutions. Other examples of symplectic resolutions include hypertoric varieties, resolutions of Slodowy slices, Hilbert schemes of points on $\mathbb{C}^2$, and, more generally, Nakajima varieties [11]. Their quantum cohomologies were studied in [10], [2], [14] and [9] respectively. The stable basis in the Springer resolutions are just characteristic cycles of Verma modules up to a sign, see [5] and Remark 3.5.3 in [9], and the restriction of stable basis to fixed points is obtained in [17]. In the case of Hilbert schemes of points on $\mathbb{C}^2$, it corresponds to Schur functions if we identify the equivariant cohomology ring of Hilbert schemes with the symmetric functions, while the fixed point basis corresponds to Jack symmetric functions, see e.g. [9], [12], [13]. In this case, Shenfeld obtained the transition matrix from the stable basis to fixed point basis in [15].

To state our main Theorem, let us fix some notations. Let B be a Borel subgroup, R^+ be the roots appearing in B, and $R^- = -R^+$. Let Δ be the set of simple roots, I be a subset of Δ, and $P = P_I = \bigcup_{w \in W_I} BwB$ be the parabolic subgroup containing B corresponding to I. It is well-known that every parabolic subgroup is conjugate to some parabolic subgroup containing the fixed Borel subgroup B, which is of the form P_I for some subset I in Δ, and P_I is not conjugate to P_J if the two subsets I and J are not equal (see [16]). Let W_P the subgroup of the Weyl group W generated by the simple reflections σ_α for $\alpha \in I$, and $R_P^\pm$ be the roots in $R^\pm$ spanned by I. Let $\alpha^\vee$ be the coroot corresponding to α. Let A be a maximal torus of G contained in B, and $\mathbb{C}^*$ scales the fiber of $T^*\mathcal{P}$ by a nontrivial character $-\hbar$. Let $T = A \times \mathbb{C}^*$.

Any weight λ that vanishes on all $\alpha^\vee \in I^\vee$ determines a one-dimensional representation $\mathbb{C}_\lambda$ of P. Define a line bundle

$$\mathcal{L}_\lambda = G \times_P \mathbb{C}_\lambda$$

on G/P. Pulling it back to $T^*\mathcal{P}$, we get a line bundle on $T^*\mathcal{P}$, which will still be denoted by $\mathcal{L}_\lambda$. Let $D_\lambda := c_1(\mathcal{L}_\lambda)$. It is well-known that the fixed point set $(T^*\mathcal{P})^A$ is in one-to-one correspondence with W/W_P. The stable envelope map stab_+ will be defined in Section 2, and $\mathrm{stab}_+(\bar{y})$ is the image of the unit in $H_T^*(\bar{y})$ under the stable envelope map, where $\bar{y}$ in $H_T^*(\bar{y})$ is the fixed point in $T^*\mathcal{P}$ corresponding to yW_P. An element $y \in W$ is called minimal if its length is minimal among the elements in the coset yW_P. As y runs through the minimal elements, $\mathrm{stab}_+(\bar{y})$ form a basis in $H_T^*(T^*\mathcal{P})$ after localization, which is called the stable basis. The result we are going to prove is:

Theorem 1.1. *The quantum multiplication by D_λ in $H_T^*(T^*\mathcal{P})$ is given by:*

$$\begin{aligned}D_\lambda * \operatorname{stab}_+(\bar{y}) =& y(\lambda)\operatorname{stab}_+(\bar{y}) - \hbar \sum_{\alpha\in R^+, y\alpha\in R^-} (\lambda,\alpha^\vee)\operatorname{stab}_+(\overline{y\sigma_\alpha}) \\ &- \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\left(\operatorname{stab}_+(\overline{y\sigma_\alpha}) + \prod_{\beta\in R_P^+}\frac{\sigma_\alpha\beta}{\beta}\operatorname{stab}_+(\bar{y})\right),\end{aligned}$$

where y is a minimal representative in yW_P, and $d(\alpha)$ is defined by Equation 3.4.

Combining this and the restriction formula for stable basis ([17]), we get

Theorem 1.2. *Under the isomorphism $H^*_{G\times\mathbb{C}^*}(T^*\mathcal{P}) \simeq (\operatorname{sym}\mathfrak{t}^*)^{W_P}[\hbar]$, the operator of quantum multiplication by D_λ is given by*

$$D_\lambda * f = \lambda f + \hbar \sum_{\alpha\in R^+\setminus R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\left(\frac{\tilde{\sigma}_\alpha(f\prod_{\beta\in R_P^+}(\beta-\hbar))}{\prod_{\beta\in R_P^+}(\beta-\hbar)} - \frac{\prod_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod_{\beta\in R_P^+}\beta}f\right).$$

This shows that the quantum multiplication formula is conjugate to the one (4.6) conjectured (through private communication) by Professor Braverman.

The paper is organized as follows. In Section 2, we apply results in [9] to define the stable basis of $T^*\mathcal{P}$. In Section 3, we prove our main Theorem 1.1 by calculating the classical multiplication and purely quantum multiplication separately. In the last section, we first show how to deduce the $G\times\mathbb{C}^*$-equivariant quantum multiplication in $T^*(G/B)$ from Theorem 1.1, which is the main result of [2]. Then a similar calculation gives a proof to Theorem 1.2.

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2. Stable basis for $T^*\mathcal{P}$

In this section, we apply the construction in [9] to $T^*\mathcal{P}$.

2.1. **Fixed point sets.** It is well-known the A-fixed points of $T^*\mathcal{P}$ is in one-to-one correspondence with W/W_P. For any $y\in W$, let $\bar{y}$ denote the coset yW_P and the corresponding fixed point in $T^*\mathcal{P}$. Recall the Bruhat order $\leq$ on W/W_P is defined as follows:

$$\bar{y}\leq\bar{w} \quad \text{if} \quad ByP/P\subseteq\overline{BwP/P}.$$

2.2. **Chamber decomposition.** The cocharacters

$$\sigma:\mathbb{C}^*\to A$$

form a lattice. Let

$$\mathfrak{a}_\mathbb{R} = \operatorname{cochar}(A)\otimes_\mathbb{Z}\mathbb{R}.$$

Define the torus roots to be the A-weights occurring in the normal bundle to $(T^*\mathcal{P})^A$. Then the root hyperplanes partition $\mathfrak{a}_\mathbb{R}$ into finitely many chambers

$$\mathfrak{a}_\mathbb{R}\setminus\bigcup\alpha_i^\perp = \coprod\mathfrak{C}_i.$$

It is easy to see that in this case the torus roots are just the roots in G. Let $+$ denote the chamber such that all root in R^+ are positive on it, and $-$ the opposite chamber.

2.3. **Stable leaves.** Let $\mathfrak{C}$ be a chamber. Define the stable leaf of $\bar{y}$ by

$$\mathrm{Leaf}_{\mathfrak{C}}(\bar{y}) = \left\{ x \in T^*\mathcal{P} \,\middle|\, \lim_{z\to 0} \sigma(z)\cdot x = \bar{y} \right\},$$

where σ is any cocharacter in $\mathfrak{C}$; the limit is independent of the choice of $\sigma \in \mathfrak{C}$. In our case,

$$\mathrm{Leaf}_{+}(\bar{y}) = T^*_{B\bar{y}P/P}\mathcal{P},$$

and

$$\mathrm{Leaf}_{-}(\bar{y}) = T^*_{B^-\bar{y}P/P}\mathcal{P},$$

where B^- is the opposite Borel subgroup.

Define a partial order on W/W_P as follows:

$$\bar{w} \preceq_{\mathfrak{C}} \bar{y} \quad \text{if} \quad \overline{\mathrm{Leaf}_{\mathfrak{C}}(\bar{y})} \cap \bar{w} \neq \emptyset.$$

By the description of $\mathrm{Leaf}_+(\bar{y})$, the order $\preceq_+$ is the same as the Bruhat order on W/W_P, and $\preceq_-$ is the opposite order. Define the slope of a fixed point $\bar{y}$ by

$$\mathrm{Slope}_{\mathfrak{C}}(\bar{y}) = \bigcup_{\bar{w}\preceq_{\mathfrak{C}}\bar{y}} \mathrm{Leaf}_{\mathfrak{C}}(\bar{w}).$$

2.4. **Stable basis.** For each $\bar{y}$, define $\epsilon_{\bar{y}} = e^A(T^*_{\bar{y}}\mathcal{P})$. Here, e^A denotes the A-equivariant Euler class. Let $N_{\bar{y}}$ denote the normal bundle of $T^*\mathcal{P}$ at the fixed point $\bar{y}$. The chamber $\mathfrak{C}$ gives a decomposition of the normal bundle

$$N_{\bar{y}} = N_{\bar{y},+} \oplus N_{\bar{y},-}$$

into A-weights which are positive and negative on $\mathfrak{C}$ respectively. The sign in $\pm e(N_{\bar{y},-})$ is determined by the condition

$$\pm e(N_{\bar{y},-})|_{H^*_A(\mathrm{pt})} = \epsilon_{\bar{y}}.$$

The following theorem is the Theorem 3.3.4 in [9] applied to $T^*\mathcal{P}$.

Theorem 2.1 ([9])**.** *There exists a unique map of $H^*_T(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C}} : H^*_T((T^*\mathcal{P})^A) \to H^*_T(T^*\mathcal{P})$$

such that for any $\bar{y} \in W/W_P$, $\Gamma = \mathrm{stab}_{\mathfrak{C}}(\bar{y})$ satisfies:

(1) $\mathrm{supp}\,\Gamma \subset \mathrm{Slope}_{\mathfrak{C}}(\bar{y})$,
(2) $\Gamma|_{\bar{y}} = \pm e(N_{-,\bar{y}})$, *with sign according to $\epsilon_{\bar{y}}$,*
(3) $\Gamma|_{\bar{w}}$ *is divisible by $\hbar$, for any $\bar{w} \prec_{\mathfrak{C}} \bar{y}$,*

where $\bar{y}$ in $\mathrm{stab}_{\mathfrak{C}}(\bar{y})$ denotes the unit in $H^*_T(\bar{y})$.

Remark 2.2.

(1) The map is defined by a Lagrangian correspondence between $(T^*\mathcal{P})^A \times T^*\mathcal{P}$, hence maps middle degree to middle degree.
(2) From the characterization, the transition matrix from $\{\mathrm{stab}_{\mathfrak{C}}(\bar{y}), \bar{y} \in W/W_P\}$ to the fixed point basis is a triangular matrix with nontrivial diagonal terms. Hence, after localization, $\{\mathrm{stab}_{\mathfrak{C}}(\bar{y}), \bar{y} \in W/W_P\}$ form a basis for the cohomology, which is the **stable basis**.
(3) Theorem 4.4.1 in [9] shows that $\{\mathrm{stab}_{\mathfrak{C}}(\bar{y}), \bar{y} \in W/W_P\}$ and $\{(-1)^m \mathrm{stab}_{-\mathfrak{C}}(\bar{y}), \bar{y} \in W/W_P\}$ are dual bases, where $m = \dim G/P$.

From now on, we let $\mathrm{stab}_{\pm}(\bar{y})$ denote the stable basis in $H^*_T(T^*\mathcal{P})$, and let $\mathrm{stab}_{\pm}(y)$ denote the stable basis in $H^*_T(T^*\mathcal{B})$. We record two lemmas here, which will be important for the calculations.

Lemma 2.3 ([1])**.** *Each coset W/W_P contains exactly one element of minimal length, which is characterized by the property that it maps I into R^+.*

Lemma 2.4 ([17]). *Let y be a minimal representative of the coset yW_P. Then*

$$\mathrm{stab}_+(\bar{y})|_{\bar{w}} \equiv \begin{cases} (-1)^{l(y)+1} \dfrac{\hbar \prod\limits_{\alpha \in R^+} \alpha}{y\beta \prod\limits_{\alpha \in R_P^+} y\sigma_\beta \alpha} & \pmod{\hbar^2} \quad \text{if } \bar{w} = \overline{y\sigma_\beta} \text{ and } y\sigma_\beta < y \text{ for some } \beta \in R^+, \\ 0 & \pmod{\hbar^2} \quad \text{otherwise,} \end{cases}$$

and

$$\mathrm{stab}_-(\bar{w})|_{\bar{y}} \equiv \begin{cases} (-1)^{l(y)+1} \dfrac{\hbar \prod\limits_{\alpha \in R^+} \alpha}{y\beta \prod\limits_{\alpha \in R_P^+} y\alpha} & \pmod{\hbar^2} \quad \text{if } \bar{w} = \overline{y\sigma_\beta} \text{ and } y\sigma_\beta < y \text{ for some } \beta \in R^+, \\ 0 & \pmod{\hbar^2} \quad \text{otherwise,} \end{cases}$$

where $<$ is the Bruhat order on the Weyl group W.

3. T-equivariant quantum cohomology of $T^*\mathcal{P}$

Now we turn to the study of equivariant quantum cohomology of $T^*\mathcal{P}$. We denote $T^*\mathcal{P}$ by X in this section. Recall $D_\lambda := c_1(\mathcal{L}_\lambda)$. We are going to determine the quantum multiplication by the divisor D_λ in terms of the stable basis. It is easy to see that $y\lambda$ does not depend on the choice of representative in yW_P, since W_P fix λ.

3.1. **Preliminaries on quantum cohomology.** By definition, the operator of quantum multiplication by $\alpha \in H_T(X)$ has the following matrix elements

$$(\alpha * \gamma_1, \gamma_2) = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \langle \alpha, \gamma_1, \gamma_2 \rangle^X_{0,3,\beta},$$

where $(\cdot,\cdot)$ denotes the standard inner product on cohomology and the quantity in angle brackets is a 3-point, genus 0, degree β equivariant Gromov–Witten invariant of X.

If α is a divisor and $\beta \neq 0$, we have

$$\langle \alpha, \gamma_1, \gamma_2 \rangle^X_{0,3,\beta} = (\alpha, \beta) \langle \gamma_1, \gamma_2 \rangle^X_{0,2,\beta}.$$

Since X has a everywhere-nondegenerate holomorphic symplectic form, it is well-known that the usual non-equivariant virtual fundamental class on $\overline{M}_{g,n}(X,\beta)$ vanishes for $\beta \neq 0$. However, we can modify the standard obstruction theory so that the virtual dimension increases by 1 (see [2] or [14]). The virtual fundamental class $[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}}$ has expected dimension

$$K_X \cdot \beta + \dim X + 2 - 3 = \dim X - 1.$$

Hence the reduced virtual class has dimension $\dim X$, and for any $\beta \neq 0$,

$$[\overline{M}_{0,2}(X,\beta)]^{\mathrm{vir}} = -\hbar \cdot [\overline{M}_{0,2}(X,\beta)]^{\mathrm{red}},$$

where $\hbar$ is the weight of the symplectic form under the $\mathbb{C}^*-$action.

3.2. **Unbroken curves.** Broken curves was introduced in [14]. Let $f : C \to X$ be an A-fixed point of $\overline{M}_{0,2}(X,\beta)$ such that the domain is a chain of rational curves

$$C = C_1 \cup C_2 \cup \cdots \cup C_k,$$

with the marked points lying on C_1 and C_k respectively.

We say f is an unbroken chain if at every node $f(C_i \cap C_{i+1})$ of C, the weights of the two branches are opposite and nonzero. Note that all the nodes are fixed by A.

More generally, if (C, f) is an A-fixed point of $\overline{M}_{0,2}(X,\beta)$, we say that f is an unbroken map if it satisfies one of the three conditions:

(1) f arises from a map $f : C \to X^A$,
(2) f is an unbroken chain, or

(3) the domain C is a chain of rational curves

$$C = C_0 \cup C_1 \cup \cdots C_k$$

such that C_0 is contracted by f, the marked points lie on C_0, and the remaining components form an unbroken chain.

Broken maps are A-fixed maps that do not satisfy any of these conditions.

Okounkov and Pandharipande proved the following Theorem in Section 3.8.3 in [14].

Theorem 3.1 ([14]). *Every map in a given connected component of $\overline{M}_{0,2}(X,\beta)^A$ is either broken or unbroken. Only unbroken components contribute to the A-equivariant localization of reduced virtual fundamental class.*

3.3. **Unbroken curves in X.** Any $\alpha \in R^+ \setminus R_P^+$ defines an SL_2 subgroup $G_{\alpha^\vee}$ of G and hence a rational curve

$$C_\alpha := G_{\alpha^\vee} \cdot [P] \subset G/P \subset X.$$

This is the unique A-invariant rational curve connecting the fixed points $\bar{1}$ and $\bar{\sigma}_\alpha$, because any such rational curve has tangent weight at $\bar{1}$ in $R^- \setminus R_P^-$, and uniqueness follows from the following lemma in Section 4 in [4].

Lemma 3.2 ([4]). *Let α, β be two roots in $R^+ \setminus R_P^+$. Then $\bar{\sigma}_\alpha = \bar{\sigma}_\beta$ if and only if $\alpha = \beta$.*

If C is an A-invariant rational curve in X, C must lie in G/P, and it connects two fixed points $\bar{y}$ and $\bar{w}$. Then its y^{-1}-translate $y^{-1}C$ is still an A-invariant curve, which connects fixed points $\bar{1}$ and $\overline{y^{-1}w}$. So $y^{-1}C = C_\alpha$ for a unique $\alpha \in R^+ \setminus R_P^+$, and $\overline{y^{-1}w} = \bar{\sigma}_\alpha$. Hence the tangent weight of C at $\bar{y}$ is $-y\alpha$. In conclusion, we have

Lemma 3.3. *There are two kinds of unbroken curves C in X:*

(1) C is a multiple cover of rational curve branched over two different fixed points,
(2) C is a chain of two rational curve $C = C_0 \cup C_1$, such that C_0 is contracted to a fixed point, the two marked points lie on C_0, and C_1 is a multiple cover of rational curve branched over two different fixed points.

For any $\alpha \in \Delta \setminus I$, define $\tau(\sigma_\alpha) := \overline{B\sigma_\alpha P/P}$. Then

$$\{\tau(\sigma_\alpha) | \alpha \in \Delta \setminus I\}$$

form a basis of $H_2(X,\mathbb{Z})$. Let $\{\omega_\alpha | \alpha \in \Delta\}$ be the fundamental weights of the root system. For any $\alpha \in R^+ \setminus R_P^+$, define degree $d(\alpha)$ of α by

$$d(\alpha) = \sum_{\beta \in \Delta \setminus I} (\omega_\beta, \alpha^\vee)\tau(\sigma_\beta). \tag{3.4}$$

Lemma 3.5 ([4]). *The degree of $[C_\alpha]$ is $d(\alpha)$, and $d(\alpha) = d(w\alpha)$ for any $w \in W_P$.*

3.4. **Classical part.** We first calculate the classical multiplication by D_λ in the stable basis. Let m denote the dimension of G/P. Since $\{\mathrm{stab}_+(\bar{y})\}$ and $\{(-1)^m \mathrm{stab}_-(\bar{y})\}$ are dual bases, we only need to calculate

$$(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) = \sum_{\bar{w} \le \bar{z} \le \bar{y}} \frac{D_\lambda|_{\bar{z}} \cdot \mathrm{stab}_+(\bar{y})|_{\bar{z}} \cdot (-1)^m \mathrm{stab}_-(\bar{w})|_{\bar{z}}}{e(T_{\bar{z}}X)}. \tag{3.6}$$

This will be zero if $\bar{y} < \bar{w}$. Assume y is a minimal representative. Note that the resulting expression lies in the nonlocalized coefficient ring due to the proof of Theorem 4.4.1 in [9], and a degree count shows that it is in $H_T^2(\mathrm{pt})$. There are two cases.

3.4.1. *Case $\bar{y} = \bar{w}$.* There is only one term in the sum of the right hand side of Equation (3.6). Hence,

$$(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{y})) = \frac{D_\lambda|_{\bar{y}} \cdot \mathrm{stab}_+(\bar{y})|_{\bar{y}} \cdot (-1)^m \mathrm{stab}_-(\bar{y})|_{\bar{y}}}{e(T_{\bar{y}}X)} = y(\lambda).$$

3.4.2. *Case $\bar{y} \neq \bar{w}$.* Notice that $(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) \in H^2_T(\mathrm{pt})$, and it is 0 if $\hbar = 0$, because every term in Equation (3.6) is divisible by $\hbar$. Hence, it is a constant multiple of $\hbar$. So in Equation (3.6), only $\bar{z} = \bar{y}$ and $\bar{z} = \bar{w}$ have contribution since all other terms are divisible by $\hbar^2$. Therefore,

$$\begin{aligned}(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) &= y(\lambda)\frac{\mathrm{stab}_-(\bar{w})|_{\bar{y}}}{\mathrm{stab}_-(\bar{y})|_{\bar{y}}} + w(\lambda)\frac{\mathrm{stab}_+(\bar{y})|_{\bar{w}}}{\mathrm{stab}_+(\bar{w})|_{\bar{w}}} \\ &= y(\lambda)\frac{\hbar \text{ part of } \mathrm{stab}_-(\bar{w})|_{\bar{y}}}{\prod_{\alpha\in R^+\setminus R^+_P} y\alpha} + w(\lambda)\frac{\hbar \text{ part of } \mathrm{stab}_+(\bar{y})|_{\bar{w}}}{\prod_{\alpha\in R^+\setminus R^+_P} w\alpha},\end{aligned}$$

where the first equality follows from $\mathrm{stab}_+(\bar{y}) \cdot \mathrm{stab}_-(\bar{y})) = (-1)^m e(T_{\bar{y}}X)$.

Lemma 2.4 shows this is zero if $\bar{w} \neq \overline{y\sigma_\beta}$ for any $\beta \in R^+$ with $y\sigma_\beta < y$. However, if $\bar{w} = \overline{y\sigma_\beta}$ for such a β, then since $(-1)^{l(y\sigma_\beta)} = (-1)^{l(y)+1}$, we have

$$\begin{aligned}&(D_\lambda \cup \mathrm{stab}_+(\bar{y}), (-1)^m \mathrm{stab}_-(\bar{w})) \\ =&y(\lambda)(-1)^{l(y)+1}\frac{\hbar \prod_{\alpha\in R^+}\alpha}{y\beta\prod_{\alpha\in R^+} y\alpha} + y\sigma_\beta(\lambda)(-1)^{l(y)+1}\frac{\hbar\prod_{\alpha\in R^+}\alpha}{y\beta\prod_{\alpha\in R^+} y\sigma_\beta\alpha} \\ =&-\frac{\hbar}{y\beta}y(\lambda) + \frac{\hbar}{y\beta}y\sigma_\beta(\lambda) \\ =&-\hbar(\lambda, \beta^\vee).\end{aligned}$$

Notice that for any $\beta \in R^+$, $y\sigma_\beta < y$ is equivalent to $y\beta \in R^-$. To summarize, we get

Theorem 3.7. *Let y be a minimal representative. Then the classical multiplication is given by*

$$D_\lambda \cup \mathrm{stab}_+(\bar{y}) = y(\lambda)\,\mathrm{stab}_+(\bar{y}) - \hbar \sum_{\alpha\in R^+, y\alpha\in R^-} (\lambda, \alpha^\vee)\,\mathrm{stab}_+(\overline{y\sigma_\alpha}).$$

3.5. **Quantum part.** Let $D_\lambda *_q$ denote the purely quantum multiplication. We want to calculate

$$(-1)^m(D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\bar{w})) = -\sum_{\beta \text{ effective}} (-1)^m \hbar q^\beta (D_\lambda, \beta)(ev_*[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\bar{w})).$$

where ev is the evaluation map from $\overline{M}_{0,2}(X, \beta)$ to $X \times X$. The $-$ sign appears because the cotangent fibers have weight $-\hbar$ under the $\mathbb{C}^*-$action. Since

$$\dim[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}} = \dim X,$$

and

$$(ev_*[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\bar{w}))$$

lies in the nonlocalized coefficient ring (see Theorem 4.4.1 in [9]), the product is a constant by a degree count. Thus we can let $\hbar = 0$, i.e., we can calculate it in A-equivariant chomology. As in the classical multiplication, there are two cases depending whether the two fixed points $\bar{y}$ and $\bar{w}$ are the same or not.

3.5.1. *Case $\bar{y} \neq \bar{w}$.* By virtual localization, Theorem 3.1 and Lemma 3.3,

$$(ev_*[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\bar{w}))$$

is nonzero if and only if $\bar{w} = \overline{y\sigma_\alpha}$ for some $\alpha \in R^+ \setminus R^+_P$. Only the first kind of unbroken curves have contribution to $(ev_*[\overline{M}_{0,2}(X, \beta)]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\overline{y\sigma_\alpha}))$, and only restriction to the fixed point $(\bar{y}, \overline{y\sigma_\alpha})$ is nonzero in the localization of the product by the first and third properties of the stable basis. The A-invariant rational curve $y[C_\alpha]$ connects the two fixed points $\bar{y}$ and $\overline{y\sigma_\alpha}$, and it is the unique one. For example, if $y[C_\beta]$ is also such a curve, then $\overline{y\sigma_\alpha} = \overline{y\sigma_\beta} = \bar{w}$. Hence $\alpha = \beta$ by Lemma 3.2. Therefore,

$$\begin{aligned}(-1)^m(D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\overline{y\sigma_\alpha})) = -\sum_{k>0}&(-1)^m \hbar q^{k\cdot d(\alpha)}(D_\lambda, k\cdot d(\alpha)) \\ &(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\overline{y\sigma_\alpha})).\end{aligned}$$

Let f be an unbroken map of degree k from $C = \mathbb{P}^1$ to $y[C_\alpha]$. Then

$$\mathrm{Aut}(f) = \mathbb{Z}/k.$$

By virtual localization,

$$k(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\overline{y\sigma_\alpha})) = \frac{e(T^*_{\bar{y}}\mathcal{P})e(T^*_{\overline{y\sigma_\alpha}}\mathcal{P})e'(H^1(C, f^*TX))}{e'(H^0(C, f^*TX))}.$$

Here e' is the product of nonzero A-weights.

We record Lemma 11.1.3 from [9].

Lemma 3.8 ([9])**.** *Let A be a torus and let $\mathcal{T}$ be an A-equivariant bundle on $C = \mathbb{P}^1$ without zero weights in the fibers $\mathcal{T}_0$ and $\mathcal{T}_\infty$. Then*

$$\frac{e'(H^0(\mathcal{T} \oplus \mathcal{T}^*))}{e'(H^1(\mathcal{T} \oplus \mathcal{T}^*))} = (-1)^{\deg \mathcal{T} + rk\mathcal{T} + z} e(\mathcal{T}_0 \oplus \mathcal{T}_\infty)$$

where $z = \dim H^1(\mathcal{T} \oplus \mathcal{T}^*)^A$, i.e., z counts the number of zero weights in $H^1(\mathcal{T} \oplus \mathcal{T}^*)$.

Since

$$f^*TX = \mathcal{T} \oplus \mathcal{T}^* \quad \text{with} \quad \mathcal{T} = f^*T\mathcal{P},$$

Lemma 3.8 gives

$$\begin{aligned} k(ev_*[\overline{M}_{0,2}(X, k\cdot d(\alpha))]^{\mathrm{red}}, \mathrm{stab}_+(\bar{y}) \otimes \mathrm{stab}_-(\overline{y\sigma_\alpha})) =& \frac{e(T^*_{\bar{y}}\mathcal{P})e(T^*_{y\bar{\sigma}_\alpha}\mathcal{P})e'(H^1(C, f^*TX))}{e'(H^0(C, f^*TX))} \\ =& (-1)^{\deg \mathcal{T} + rk\mathcal{T} + z}. \end{aligned}$$

We now study the vector bundle $\mathcal{T} = f^*T\mathcal{P}$. First of all, $rk\mathcal{T} = \dim \mathcal{P}$. By localization,

$$\begin{aligned} \deg \mathcal{T} &= k \left(\frac{\sum_{\gamma \in R^+ \setminus R^+_P} (-y\gamma)}{-y\alpha} + \frac{\sum_{\gamma \in R^+ \setminus R^+_P} (-y\sigma_\alpha\gamma)}{y\alpha} \right) \\ &= k \sum_{\gamma \in R^+ \setminus R^+_P} (\gamma, \alpha^\vee) = k(2\rho - 2\rho_P, \alpha^\vee) \\ &= 2k \sum_{\beta \in \Delta \setminus I} (\omega_\beta, \alpha^\vee) \end{aligned}$$

is an even number, where ρ is the half sum of the positive roots, ρ_P is the half sum of the positive roots in R^+_P, and ω_β are the fundamental weights.

The vector bundle $\mathcal{T}$ splits as a direct sum of line bundles on C

$$\mathcal{T} = \bigoplus_i \mathcal{L}_i,$$

so

$$\bigoplus_i \mathcal{L}_i|_0 = \bigoplus_{\gamma \in R^+ \setminus R^+_P} \mathfrak{g}_{-y\gamma},$$

where $\mathfrak{g}_{-y\gamma}$ are the root subspaces of $\mathfrak{g}$. Suppose $\mathcal{L}_i|_0 = \mathfrak{g}_{-y\gamma}$. Since $y\sigma_\alpha y^{-1}$ maps y to $y\sigma_\alpha$, we have

$$\mathcal{L}_i|_\infty = \mathfrak{g}_{-y\sigma_\alpha\gamma}.$$

Hence there is only one zero weight in $H^1(\mathcal{T} \oplus \mathcal{T}^*)$, which occurs in $H^1(\mathcal{L}_i \oplus \mathcal{L}_i^*)$, where $\mathcal{L}_i|_0 = \mathfrak{g}_{-y\alpha}$, i.e., $\mathcal{L}_i$ is the tangent bundle of C.

Therefore $z = 1$ and we have

Lemma 3.9.

$$(-1)^m (D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\overline{y\sigma_\alpha})) = \sum_{k>0} \hbar q^{k\cdot d(\alpha)} (D_\lambda, d(\alpha)) = -\hbar \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} (\lambda, \alpha^\vee).$$

Proof. We only need to show

$$(D_\lambda, d(\alpha)) = -(\lambda, \alpha^\vee).$$

By definition and localization,

$$\begin{aligned}(D_\lambda, d(\alpha)) &= \sum_{\beta\in\Delta\setminus I} (\omega_{\beta,\alpha^\vee}) \int_{\tau(\sigma_\beta)} c_1(\mathcal{L}_\lambda) = \sum_{\beta\in\Delta\setminus I} (\omega_{\beta,\alpha^\vee}) \left(\frac{\lambda}{-\beta} + \frac{\sigma_\beta\lambda}{\beta}\right) \\ &= -\sum_{\beta\in\Delta\setminus I} (\omega_{\beta,\alpha^\vee})(\lambda, \beta^\vee) = -\sum_{\beta\in\Delta} (\omega_{\beta,\alpha^\vee})(\lambda, \beta^\vee) \\ &= -(\lambda, \alpha^\vee).\end{aligned}$$

□

3.5.2. *Case* $\bar{y} = \bar{w}$. In this case, only the second kind of unbroken curves have contribution to $(D_\lambda *_q \mathrm{stab}_+(\bar{y}), \mathrm{stab}_-(\bar{y}))$. Let $C = C_0 \cup C_1$ be an unbroken curve of the second kind with C_0 contracted to the fixed point $\bar{y}$, and C_1 is a cover of the rational curve yC_α of degree k, where $\alpha \in R^+ \setminus R_P^+$. Let p denote the node of C, and let f be the map from C to X. Then the corresponding decorated graph Γ has two vertices, one of them has two marked tails, and there is an edge of degree k connecting the two vertices. Hence the automorphism group of the graph is trivial. The virtual normal bundle ([6]) is

$$e(N_\Gamma^{\mathrm{vir}}) = \frac{e'(H^0(C, f^*TX))}{e'(H^1(C, f^*TX))} \frac{-y\alpha/k}{y\alpha/k} = -\frac{e'(H^0(C, f^*TX))}{e'(H^1(C, f^*TX))}, \tag{3.10}$$

where $e'(H^0(C, f^*TX))$ denotes the nonzero A-weights in $H^0(C, f^*TX)$. Consider the normalization exact sequence resolving the node of C:

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_1} \to \mathcal{O}_p \to 0.$$

Tensoring with f^*TX and taking cohomology yields:

$$\begin{aligned}0 \to H^0(C, f^*TX) \to H^0(C_0, f^*TX) \oplus H^0(C_1, f^*TX) \to T_{\bar{y}}X \\ \to H^1(C, f^*TX) \to H^1(C_0, f^*TX) \oplus H^1(C_1, f^*TX) \to 0.\end{aligned}$$

Since C_0 is contracted to $\bar{y}$, $H^0(C_0, f^*TX) = T_{\bar{y}}X$ and $H^1(C_0, f^*TX) = 0$. Therefore, as virtual representations, we have

$$H^0(C, f^*TX) - H^1(C, f^*TX) = H^0(C_1, f^*TX) - H^1(C_1, f^*TX).$$

Due to Equation (3.10) and the analysis in the last case, we get

$$\begin{aligned}e(N_\Gamma^{\mathrm{vir}}) &= -\frac{e'(H^0(C_1, f^*TX))}{e'(H^1(C_1, f^*TX))} \\ &= (-1)^m e(T_{\bar{y}}\mathcal{P}) e(T_{\overline{y\sigma_\alpha}}\mathcal{P}).\end{aligned}$$

Then by virtual localization formula, we have

$$
\begin{aligned}
(-1)^m(D_\lambda *_q \operatorname{stab}_+(\bar{y}), \operatorname{stab}_-(\bar{y})) &= -\hbar \sum_{\alpha\in R^+\setminus R_P^+, k>0} (D_\lambda, d(\alpha)) q^{k\cdot d(\alpha)} \frac{e(T^*_{\bar{y}}\mathcal{P})^2}{e(T_{\bar{y}}\mathcal{P})e(T_{\overline{y\sigma_\alpha}}\mathcal{P})} \\
&= \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod_{\beta\in R^+\setminus R_P^+} y\beta}{\prod_{\beta\in R^+\setminus R_P^+} y\sigma_\alpha\beta} \\
&= \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod_{\beta\in R^+} y\beta}{\prod_{\beta\in R^+} y\sigma_\alpha\beta} \frac{\prod_{\beta\in R_P^+} y\sigma_\alpha\beta}{\prod_{\beta\in R_P^+} y\beta} \\
&= -\hbar\cdot y \left(\sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod_{\beta\in R_P^+} \beta} \right).
\end{aligned}
$$

Here we have used

$$
\prod_{\beta\in R^+} y\beta = (-1)^{l(y)} \prod_{\beta\in R^+} \beta, \quad \text{and} \quad (-1)^{l(y\sigma_\alpha)} = (-1)^{l(y)+l(\sigma_\alpha)} = (-1)^{l(y)+1}.
$$

Notice that for any root $\gamma\in R_P^+$, σ_γ preserves $R^+\setminus R_P^+$. For any $\alpha\in R^+\setminus R_P^+$, $d(\sigma_\gamma(\alpha)) = d(\alpha)$, $(\lambda,\alpha^\vee) = (\lambda, \sigma_\gamma(\alpha)^\vee)$ and $\prod_{\beta\in R_P^+} \sigma_\gamma\beta = -\prod_{\beta\in R_P^+}\beta$. Hence,

$$
\begin{aligned}
&\sigma_\gamma\left(\sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_\alpha\beta \right) \\
&\quad = \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_{\sigma_\gamma\alpha}\sigma_\gamma\beta \\
&\quad = -\sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_\alpha\beta.
\end{aligned}
$$

Therefore $\sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \prod_{\beta\in R_P^+} \sigma_\alpha\beta$ is divisible by $\prod_{\beta\in R_P^+}\beta$. But they have the same degree, so

$$
\sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod_{\beta\in R_P^+} \beta} \tag{3.11}
$$

is a scalar.

To summarize, we get

Theorem 3.12. *The purely quantum multiplication by D_λ in $H_T^*(T^*\mathcal{P})$ is given by:*

$$
D_\lambda *_q \operatorname{stab}_+(\bar{y}) = -\hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \operatorname{stab}_+(\overline{y\sigma_\alpha}) - \hbar \sum_{\alpha\in R^+\setminus R_P^+} (\lambda,\alpha^\vee) \frac{q^{d(\alpha)}}{1-q^{d(\alpha)}} \frac{\prod_{\beta\in R_P^+} \sigma_\alpha\beta}{\prod_{\beta\in R_P^+} \beta} \operatorname{stab}_+(\bar{y}).
$$

Remark 3.13.

(1) The scalar

$$-\hbar \sum_{\alpha\in R^+\setminus R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod\limits_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+}\beta}$$

can also be determined by the condition

$$D_\lambda *_q 1 = 0.$$

(2) The element y is not necessarily a minimal representative.

(3) The Theorem is also true if we replace all the stab_+ by stab_-.

3.6. **Quantum multiplications.** Combining Theorem 3.7 and Theorem 3.12, we get our main Theorem 1.1. Taking $I=\emptyset$, we get the quantum multiplication by D_λ in $H_T^*(T^*\mathcal{B})$.

Theorem 3.14. *The quantum multiplication by D_λ in $H_T^*(T^*\mathcal{B})$ is given by:*

$$\begin{aligned}D_\lambda * \mathrm{stab}_+(y) =& y(\lambda)\,\mathrm{stab}_+(y) - \hbar \sum_{\alpha\in R^+, y\alpha\in -R^+}(\lambda,\alpha^\vee)\,\mathrm{stab}_+(y\sigma_\alpha)\\ &-\hbar\sum_{\alpha\in R^+}(\lambda,\alpha^\vee)\frac{q^{\alpha^\vee}}{1-q^{\alpha^\vee}}(\mathrm{stab}_+(y\sigma_\alpha)+\mathrm{stab}_+(y)).\end{aligned}$$

3.7. **Calculation of the scalar in type A.** We can define an equivalence relation on $R^+\setminus R_P^+$ as follows

$$\alpha\sim\beta \quad \text{if} \quad d(\alpha)=d(\beta).$$

Then $w(\alpha)\sim\alpha$ for any $w\in W_P$. We have

$$\begin{aligned}&\sum_{\alpha\in R^+\setminus R_P^+}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod\limits_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+}\beta}\\ =&\sum_{\alpha\in(R^+\setminus R_P^+)/\sim}(\lambda,\alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\sum_{\alpha'\sim\alpha}\frac{\prod\limits_{\beta\in R_P^+}\sigma_{\alpha'}\beta}{\prod\limits_{\beta\in R_P^+}\beta}.\end{aligned}$$

It is easy to see that

$$\sum_{\alpha'\sim\alpha}\frac{\prod\limits_{\beta\in R_P^+}\sigma_{\alpha'}\beta}{\prod\limits_{\beta\in R_P^+}\beta}$$

is a constant, which will be denoted by $C_P(\alpha)$.

In this section, we will determine the constant $C_P(\alpha)$ when G is of type A. We will first calculate this number in $T^*Gr(k,n)$ case, and the general case will follow easily. Now let $G=SL(n,\mathbb{C})$ and let x_i be the function on the Lie algebra of the diagonal torus defined by $x_i(t_1,\cdots,t_n)=x_i$.

3.7.1. *$T^*Gr(k,n)$ case.* Let P be a parabolic subgroup containing the upper triangular matrices such that $T^*(G/P)$ is $T^*Gr(k,n)$. Then

$$R_P^+=\{x_i-x_j|1\le i<j\le k, \text{ or } k<i<j\le n\},\quad R\setminus R_P^+=\{x_i-x_j|1\le i\le k<j\le n\}$$

and all the roots in $R\setminus R_P^+$ are equivalent. The number $C_P(\alpha)$ will be denoted by C_P. By definition,

$$C_P=\frac{\sum\limits_{1\le r\le k<s\le n}(rs)\left(\prod\limits_{1\le i<j\le k}(x_i-x_j)\prod\limits_{1+k\le p<q\le n}(x_p-x_q)\right)}{\prod\limits_{1\le i<j\le k}(x_i-x_j)\prod\limits_{1+k\le p<q\le n}(x_p-x_q)},\tag{3.15}$$

where (rs) means the transposition of x_r and x_s.

Observe that

$$\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)=\det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{k-1} & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 1 & x_k & \cdots & x_k^{k-1} & & & & \\ & & & & 1 & x_{k+1} & \cdots & x_{k+1}^{n-k-1} \\ & & & & \vdots & \vdots & \ddots & \\ & & & & 1 & x_n & \cdots & x_n^{n-k-1} \end{pmatrix}.$$

Then it is easy to see that the coefficient of $x_2x_3^2\cdots x_k^{k-1}x_{k+2}x_{k+3}^2\cdots x_n^{n-k-1}$ in

$$\sum_{1\le r\le k<s\le n}(rs)\left(\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)\right)$$

is $\min(k,n-k)$, since only when $s-r=k$, $(rs)\left(\prod_{1\le i<j\le k}(x_j-x_i)\prod_{1+k\le p<q\le n}(x_q-x_p)\right)$ has the term $x_2x_3^2\cdots x_k^{k-1}x_{k+2}x_{k+3}^2x_n^{n-k-1}$, and the coefficient is 1. Hence

Proposition 3.16.

$$C_P=\min(k,n-k).$$

3.7.2. *General case.* Let $\lambda=(\lambda_1,\cdots,\lambda_N)$ be a partition of n with $\lambda_1\ge\cdots\ge\lambda_N$. Let

$$\mathcal{F}_\lambda=\{0\subset V_1\subset V_2\cdots\subset V_N|\dim V_i/V_{i-1}=\lambda_i\}$$

be the partial flag variety, and let P be the corresponding parabolic subgroup. Then

$$R_P^+=\{x_i-x_j|\lambda_1+\cdots+\lambda_p<i<j\le\lambda_1+\cdots+\lambda_{p+1},\ \text{for some } p \text{ between } 0 \text{ and } N-1\}.$$

Two positive roots x_i-x_j and x_k-x_l are equivalent if and only if there exist $1\le p<q\le N$ such that

$$\lambda_1+\cdots+\lambda_p<i,k\le\lambda_1+\cdots+\lambda_{p+1},\lambda_1+\cdots+\lambda_q<j,l\le\lambda_1+\cdots+\lambda_{q+1}.$$

So the set $(R^+\setminus R_P^+)/\sim$ has representatives

$$\{x_{\lambda_1+\cdots+\lambda_p}-x_{\lambda_1+\cdots+\lambda_q}|1\le p<q\le N\}.$$

The same analysis as in the last case gives

Proposition 3.17. *For any* $1\le p<q\le N$,

$$C_P(x_{\lambda_1+\cdots+\lambda_p}-x_{\lambda_1+\cdots+\lambda_q})=\lambda_q.$$

4. $G\times\mathbb{C}^*$ quantum multiplications

Let $\mathbb{G}=G\times\mathbb{C}^*$, and let $\mathcal{B}$ denote the flag variety G/B. In this section, we will first get the $\mathbb{G}$-equivariant quantum multiplication formula in $T^*\mathcal{B}$, which is the main result of [2]. Then we show the quantum multiplication formula in $T^*\mathcal{P}$ is conjugate to the conjectured formula given by Braverman .

4.1. $T^*\mathcal{B}$ **case.** Let us recall the result from [2] first. Let $\mathfrak{t}$ be the Lie algebra of the maximal torus A. Then

$$H^*_{\mathbb{G}}(T^*\mathcal{B})\simeq H^*_T(T^*\mathcal{B})^W\simeq H^*_T(\mathrm{pt})\simeq \operatorname{sym}\mathfrak{t}^*[\hbar].$$

The isomorphism is determined as follows: for any $\beta\in H^*_{\mathbb{G}}(T^*\mathcal{B})$, lift it to $H^*_T(T^*\mathcal{B})$, and then restrict it to the fixed point 1. Similarly, we have

$$H^*_{\mathbb{G}}(T^*\mathcal{P})\simeq H^*_T(T^*\mathcal{P})^W\simeq(\operatorname{sym}\mathfrak{t}^*)^{W_P}[\hbar].$$

Let us recall the definition of the graded affine Hecke algebra $\mathcal{H}_\hbar$. It is generated by the symbols x_λ for $\lambda\in\mathfrak{t}^*$, Weyl elements $\bar{w}$ and a central element $\hbar$ such that

(1) x_λ depends linearly on $\lambda\in\mathfrak{t}^*$;
(2) $x_\lambda x_\mu=x_\mu x_\lambda$;
(3) the $\tilde{w}$'s form the Weyl group inside $\mathcal{H}_t$;

(4) for any $\alpha \in \Delta$, $\lambda \in \mathfrak{t}^*$, we have

$$\tilde{\sigma}_\alpha x_\lambda - x_{\tilde{\sigma}_\alpha(\lambda)}\tilde{\sigma}_\alpha = \hbar(\alpha^\vee, \lambda).$$

According to [8], we have a natural isomorphism

$$H_*^{\mathbb{G}}(T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}) \simeq \mathcal{H}_\hbar,$$

where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. The action of $\mathcal{H}_\hbar$ on $\operatorname{sym}\mathfrak{t}^*[\hbar]$ is defined as follows: x_λ acts by multiplication by λ, and for every simple root α, the action of $\tilde{\sigma}_\alpha$ is defined by

$$\tilde{\sigma}_\alpha f = (\frac{\hbar}{\alpha} + \frac{\alpha - \hbar}{\alpha}\sigma_\alpha)f$$

where $f \in \operatorname{sym}\mathfrak{t}^*[\hbar]$, and $\sigma_\alpha f$ is the usual Weyl group action on $\operatorname{sym}\mathfrak{t}^*[\hbar]$.

Having introduced the above notations, we can state the main Theorem of [2].

Theorem 4.1 ([2]). *The operator of quantum multiplication by D_λ in $H_{\mathbb{G}}^*(T^*\mathcal{B})$ is equal to*

$$x_\lambda + \hbar \sum_{\alpha \in R^+} (\lambda, \alpha^\vee)\frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}}(\tilde{\sigma}_\alpha - 1).$$

Let us also recall the restriction formula for stable basis from [17].

Theorem 4.2. *Let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y \in W$, and $w \leq y$. Then*

$$\operatorname{stab}_+(y)|_w = \sum_{\substack{1\leq i_1<i_2<\cdots<i_k\leq l\\ w=\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}}} (-1)^l \prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j} - \hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}} \frac{\hbar^{l-k}}{\prod_{j=0}^{k}\prod_{i_j<r<i_{j+1}} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r} \prod_{\alpha \in R^+} \alpha,$$

where σ_i is the simple reflection associated to a simple root α_i.

We are now ready to deduce Theorem 4.1 from Theorem 3.14 and Theorem 4.2. The classical multiplication is obvious. We only show that the purely quantum part matches. Let $f \in \operatorname{sym}\mathfrak{t}^*[\hbar]$ correspond to $\gamma \in H_{\mathbb{G}}^*(T^*\mathcal{B})$. We also let γ denote the lift in $H_T^*(T^*\mathcal{B})$. Then $\gamma|_w = w(f)$ for any $w \in W$. Since the stable and unstable basis are dual basis up to $(-1)^n$, where $n = \dim\mathcal{B}$, we have

$$\gamma = \sum_y (-1)^n(\gamma, \operatorname{stab}_+(y))\operatorname{stab}_-(y).$$

Due to Theorem 3.14, we have

$$D_\lambda *_q \gamma = -\hbar \sum_{\alpha \in R^+} (\lambda, \alpha^\vee)\frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} \sum_y (\gamma, (-1)^n \operatorname{stab}_+(y))(\operatorname{stab}_-(y\sigma_\alpha) + \operatorname{stab}_-(y)).$$

Notice that $\operatorname{stab}_-(y)|_1 = \delta_{y,1}e(T_1^*\mathcal{B})$. Restricting to the fixed point 1 , we get

$$\begin{aligned} D_\lambda *_q \gamma|_1 = &-\hbar \sum_{\alpha \in R^+} (\lambda, \alpha^\vee)\frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}}\gamma|_1 \\ &- \hbar \sum_{\alpha \in R^+} (\lambda, \alpha^\vee)\frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}}(\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha))e(T_1^*\mathcal{B}). \end{aligned}$$

Hence we only need to show

$$-(\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha))e(T_1^*\mathcal{B}) = \tilde{\sigma}_\alpha f. \tag{4.3}$$

To prove this, we need the following lemma.

Lemma 4.4. *If $w = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$, then*

$$\prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_{j-1}}\alpha_{i_j} - \hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j} - \hbar} = \frac{e(T_1^*\mathcal{B})}{e(T_w^*\mathcal{B})}.$$

Proof. If $w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$ is reduced, then this follows from the fact

$$\{w\beta | \beta \in R^+, w\beta \in R^-\} = \{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_j}\alpha_{i_j} | 1 \le j \le l\}.$$

If $w = (\sigma_\alpha\sigma_\beta)^{m(\alpha,\beta)} = 1$ for some simple roots α and β, where $m(\alpha,\beta)$ is the order of $\sigma_\alpha\sigma_\beta$, we can check it case by case easily. If $w = \sigma_\alpha^2$, then it it trivial. In general, w will be a composition of these three cases. $\square$

If $\sigma_\alpha = \sigma_{\alpha_1}\cdots\sigma_{\alpha_l}$ is a reduced decomposition, then

$$\tilde{\sigma}_\alpha f = \prod_{i=1}^{l}\left(\frac{\hbar}{\alpha_i} + \frac{\alpha_i - \hbar}{\alpha_i}\sigma_{\alpha_i}\right)f.$$

Expanding this and using Theorem 4.2, Lemma 4.4 and the fact $(-1)^{l(\sigma_\alpha)} = -1$, we get

$$\tilde{\sigma}_\alpha(f) = \sum_w \frac{\operatorname{stab}_+(\sigma_\alpha)|_w wf}{e(T_wT^*\mathcal{B})}(-1)^{1+n}e(T_1^*\mathcal{B}) = -(\gamma, (-1)^n \operatorname{stab}_+(\sigma_\alpha))e(T_1^*\mathcal{B}), \tag{4.5}$$

which is precisely Equation (4.3).

4.2. $T^*\mathcal{P}$ **case.** In the parabolic case, Professor Braverman suggests (through private communication) that the quantum multiplication should be

$$D_\lambda * = x_\lambda + \hbar \sum_{\alpha\in R^+\setminus R_P^+}(\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\tilde{\sigma}_\alpha + \cdots, \tag{4.6}$$

where $\cdots$ is some scalar. Recall we have

$$H^*_{\mathbb{G}}(T^*\mathcal{P}) \simeq H^*_T(T^*\mathcal{P})^W \simeq (\operatorname{sym}\mathfrak{t}^*)^{W_P}[\hbar].$$

It is easy to see that classical multiplication by D_λ is given by multiplication by λ.

Now we do the similar calculation as in the $T^*\mathcal{B}$ case. We need the following restriction formula from [17]:

$$\operatorname{stab}_\pm(\bar{y})|_{\bar{w}} = \sum_{\bar{z}=\bar{w}} \frac{\operatorname{stab}_\pm(y)|_z}{\prod\limits_{\alpha\in R_P^+} z\alpha}. \tag{4.7}$$

Take any $\gamma \in H^*_{\mathbb{G}}(T^*\mathcal{P})$, and assume it corresponds to $f \in (\operatorname{sym}\mathfrak{t}^*)^{W_P}[\hbar]$. We still let γ denote the corresponding lift in $H^*_T(T^*\mathcal{P})$. Then $\gamma|_{\bar{y}} = yf$. Let m be the dimension of $\mathcal{P}$. Then we have

$$\gamma = \sum_{\bar{y}}(-1)^m(\gamma, \operatorname{stab}_+(\bar{y}))\operatorname{stab}_-(\bar{y}).$$

By Theorem 3.12,

$$\begin{aligned} D_\lambda *_q \gamma = &\sum_{\bar{y}}(\gamma, (-1)^m \operatorname{stab}_+(\bar{y}))(-\hbar)\sum_{\alpha\in R^+\setminus R_P^+}(\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\operatorname{stab}_-(\overline{y\sigma_\alpha}) \\ &- \hbar\sum_{\alpha\in R^+\setminus R_P^+}(\lambda, \alpha^\vee)\frac{q^{d(\alpha)}}{1-q^{d(\alpha)}}\frac{\prod\limits_{\beta\in R_P^+}\sigma_\alpha\beta}{\prod\limits_{\beta\in R_P^+}\beta}\gamma. \end{aligned}$$

Notice that

$$\operatorname{stab}_-(\overline{y\sigma_\alpha})|_{\bar{1}} = \begin{cases} e(T_{\bar{1}}^*\mathcal{P}) & \text{if } \overline{y\sigma_\alpha} = \bar{1}; \\ 0 & \text{otherwise}. \end{cases}$$

Restricting $D_\lambda *_q \gamma$ to the fixed point $\bar{1}$, we get

$$D_\lambda *_q \gamma|_{\bar{1}} = -\hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} (\gamma, (-1)^m \operatorname{stab}_+(\bar{\sigma}_\alpha)) e(T_{\bar{1}}^* \mathcal{P})$$
$$- \hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} \frac{\prod_{\beta \in R_P^+} \sigma_\alpha \beta}{\prod_{\beta \in R_P^+} \beta} f$$

Due to restriction formula (4.7) and Equation (4.5), we have

$$(\gamma, (-1)^m \operatorname{stab}_+(\bar{\sigma}_\alpha)) e(T_{\bar{1}}^* \mathcal{P}) = - \frac{\tilde{\sigma}_\alpha (f \prod_{\beta \in R_P^+} (\beta - \hbar))}{\prod_{\beta \in R_P^+} (\beta - \hbar)}.$$

Hence, we obtain Theorem 1.2.

Since

$$\hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}} \frac{\prod_{\beta \in R_P^+} \sigma_\alpha \beta}{\prod_{\beta \in R_P^+} \beta} \tag{4.8}$$

is a scalar, the quantum multiplication formula in Theorem 1.2 is conjugate to the conjectured formula (4.6) by the function

$$\prod_{\beta \in R_P^+} (\beta - \hbar).$$

This factor comes from geometry as follows. Let π be the projection map from $\mathcal{B}$ to $\mathcal{P}$, and Γ_π be its graph. Then the conormal bundle to Γ_π in $\mathcal{B} \times \mathcal{P}$ is a Lagrangian submanifold of $T^*(\mathcal{B} \times \mathcal{P})$.

$$\begin{tikzcd} T^*_{\Gamma_\pi}(\mathcal{B} \times \mathcal{P}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & T^*\mathcal{B} . \\ T^*\mathcal{P} & \end{tikzcd}$$

Let $D = p_{1*} p_2^*$ be the map from $H^*_{\mathbb{G}}(T^*\mathcal{P})$ to $H^*_{\mathbb{G}}(T^*\mathcal{B})$ induced by this correspondence. Then under the isomorphisms

$$H^*_{\mathbb{G}}(T^*\mathcal{B}) \simeq \operatorname{sym} \mathfrak{t}^*[\hbar] \quad \text{and} \quad H^*_{\mathbb{G}}(T^*\mathcal{P}) \simeq (\operatorname{sym} \mathfrak{t}^*)^{W_P}[\hbar],$$

the map becomes multiplicaiton by the above factor, see [17]. The scalar in the conjectured formula (4.6) is just the one in Equation (4.8). By the calculation in the Subsection 3.7, it is not equal to

$$\hbar \sum_{\alpha \in R^+ \setminus R_P^+} (\lambda, \alpha^\vee) \frac{q^{d(\alpha)}}{1 - q^{d(\alpha)}}$$

in general. It can also be determined by the condition $D_\lambda *_q 1 = 0$.

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